MATH4060 Solution 2

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Exercise 5

Similar to arguments in the previous homework, we observe that F_α is entire because $\alpha > 1$ enables us to exhibit $F_\alpha(z)$ as the uniform limit of $\int_{-n}^{n} e^{-|t|^\alpha} e^{2\pi i z t}\, dt$ (as $n \to \infty$) in every horizontal strip S_b. To prove that F_α has growth order $\leq \alpha/(\alpha-1)$, we first show that there is some constant $c > 0$ such that

$$-\frac{|t|^\alpha}{2} + 2\pi|z||t| \leq c|z|^{\alpha/(\alpha-1)} \tag{1}$$

for any $t \in \mathbb{R}$ and $z \in \mathbb{C}$. This is in fact a consequence of Young's inequality, but we can derive it more directly as follows: Let $\alpha' = \alpha/(\alpha-1)$. Note that for $r, s \geq 0$, we have $rs \leq r^\alpha + s^{\alpha'}$, because if $r^\alpha \geq rs$, then we are done, otherwise $r^\alpha < rs$ and so $r^{\alpha-1} < s$ and $r = r^{(\alpha-1)(\alpha'-1)} < s^{\alpha'-1}$. Multiplying by s shows that $rs < s^{\alpha'}$. Now take $r = |t|$ and $s = 4\pi|z|$.

Using (1), we compute that

$$|F_\alpha(z)| \leq \int_{-\infty}^{\infty} e^{-|t|^\alpha/2} e^{-|t|^\alpha/2 + 2\pi|z||t|}\, dt \leq e^{c|z|^{\alpha/(\alpha-1)}} \int_{-\infty}^{\infty} e^{-|t|^\alpha/2}\, dt = Ce^{c|z|^{\alpha/(\alpha-1)}}.$$

So the growth order of F_α is $\leq \alpha/(\alpha-1)$. On the other hand, take $z = -ix$, $x > 0$ to see that for any $b > a > 0$,

$$|F_\alpha(-ix)| = \int_{-\infty}^{\infty} e^{-|t|^\alpha} e^{2\pi x t}\, dt \geq \int_a^b e^{-t^\alpha} e^{2\pi x t}\, dt \geq (b-a)e^{-b^\alpha} e^{2\pi x a}.$$

Setting $b = x^{1/(\alpha-1)}$ and $a = x^{1/(\alpha-1)} - 1$ shows that

$$|F_\alpha(-ix)| \geq e^{(2\pi-1)x^{\alpha/(\alpha-1)} - 2\pi x} \geq C' e^{c' x^{\alpha/(\alpha-1)}}$$

for some constant $C', c' > 0$ (independent of x) because $\alpha/(\alpha-1) > 1$. Then take $x \to \infty$ to see that the growth order of F_α is at least $\alpha/(\alpha-1)$.

Exercise 6

Taking $z = 1/2$ in the product formula $\sin \pi z/\pi = z \prod_{m=1}^{\infty}(1 - z^2/m^2)$, we have

$$\frac{1}{\pi} = \frac{1}{2} \prod_{m=1}^{\infty} \left(1 - \frac{1}{4m^2}\right) = \frac{1}{2} \prod_{m=1}^{\infty} \frac{(2m-1)(2m+1)}{4m^2}.$$

Exercise 9

Note that for $|z| < 1$ fixed,

$$(1-z) \prod_{k=0}^{N} (1 + z^{2^k}) = 1 - z^{2^{N+1}} \to 1$$

as $N \to \infty$, so $(1-z) \prod_{k=0}^{\infty}(1 + z^{2^k}) = 1$ as required.

Exercise 13

It is easy to see that $e^z - z$ has growth order 1. So if $e^z - z$ has only finitely many zeros, we can use Hadamard's theorem to write $e^z - z = e^{az}p(z)$ for some constant a and polynomial $p(z)$. Note that a is nonzero (otherwise $e^z = p(z) + z$). Taking derivative twice, we have $e^{(a-1)z}(a^2p(z) + 2ap'(z) + p''(z)) = 1$. By the fundamental theorem of algebra, since $a^2 \neq 0$, $p(z)$ must be a constant, say $p(z) = b$. But $a^2be^{(a-1)z} = 1$ implies that $a = b = 1$. So $e^z - z = e^{az}p(z) = e^z$, which is impossible.

Note that one can show $e^z - q(z) = 0$ has infinitely many solutions for any nonzero polynomial $q(z)$ by the same argument (but taking derivative $\deg(q) + 1$ times).